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Improved Gell-Mann–Okubo Relations and SU(3) Rotation Excitations of Baryon States*

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The corrections to the Gell-Mann–Okubo relations of baryon masses are presented in the SU(3) Skyrme model. These corrections are calculated up to the second order in flavor breaking at the skyrmion quantum mechanics. The results are compatible with the experimental data. They could be regarded as a signal of existence of the SU(3) rotation excitation states of baryons: 27-let (with spin 1/2 or 3/2), 10*-let (with spin 1/2) and 35-let (with spin 3/2).

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The Gell-Mann–Okubo relations (GOR) of baryon masses[1] were originally formulated in terms of a perturbative treatment of flavor breaking in SU(3) group theory. In the history, the success of GOR (or SU(3) theory) led to the quark conception and deep understanding for the strong interactions. In recent years a new (GOR type) baryon mass formula including octet and decuplet masses has been deduced[2] using only general properties of QCD. Also some effective theories of QCD derived[3] new mass formulas improving GOR. According to GOR, the baryon-octet and baryon-decuplet mass relations can be written as

$$2(M_N + M_\Xi) = 3M_\Lambda + M_\Sigma + \delta m_8, \tag{1}$$

$$M_\Delta - M_{\Sigma^*} = M_{\Sigma^*} - M_{\Xi^*} + \delta m_{10}^{(1)} = M_{\Xi^*} - M_\Omega + \delta m_{10}^{(2)}. \tag{2}$$

Here as $\delta m_8 = \delta m_{10}^{(1)} = \delta m_{10}^{(2)} = 0$, equations (1) and (2) are the standard GOR. However, in the real world

$$\delta m_8 = -26MeV, \quad \delta m_{10}^{(1)} = 4.5MeV, \quad \delta m_{10}^{(2)} = 9.6MeV, \tag{3}$$

which stand for the deviations of GOR from experimental data. The motivation of our studies in the present paper is to calculate δm_8, $\delta m_{10}^{(1)}$ and $\delta m_{10}^{(2)}$ in terms of the Skyrme model[4, 5] analytically.

In the Skyrme model the baryon octet and baryon decuplet emerge as topological solitons (i.e., skyrmions) in the SU(3)×SU(3) current algebraic chiral Lagrangians. It is believed that this chiral soliton model provides a reasonable dynamical mechanism for the mass splitting of SU(3) baryons. In actual factor, through calculations of the first order perturbation to the masses of baryons in the SU(3) Skyrme model, one can re-establish GOR[5]. Thus it could be expected that the high order correction calculations in the perturbations of the skyrmion quantum mechanics will show the the values of δm_8, $\delta m_{10}^{(1)}$ and $\delta m_{10}^{(2)}$. In the present paper we will complete the calculations of the second order corrections which should be the leading order to δm_8, $\delta m_{10}^{(1)}$ and $\delta m_{10}^{(2)}$, and learn some new physical implications from them.

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The Hamiltonian of the SU(3) skyrmion quantum mechanics in the SU(3) collective coordinate space is[6] (we use the notations of ref.[6] hereafter)

$$
\begin{aligned}
H &= H_0 + H', && (4)\\
H_0 &= M_s + \frac{1}{2b^2}(\sum_{i=1}^{8} L_i L_i - R_8^2) + \frac{1}{2}(\frac{1}{a^2} - \frac{1}{b^2})\sum_{A=1}^{3} R_A R_A + \frac{2\delta}{\sqrt{3}} F_\pi R_8, && (5)\\
H' &= m[1 - D_{88}^{(ad)}(A)], && (6)
\end{aligned}
$$

where M_s is a classical soliton's mass, a^2 and b^2 are the soliton's moments of inertia, m, F_π and δ are constants and parameters in the model, $D_{88}^{(ad)}(A)$ denotes the regular adjoint representation functions of SU(3), and $[L_i, L_j] = i f_{ijk} L_k$, $[R_i, R_j] = -i f_{ijk} R_k$, $[L_i, R_j] = 0$. In Eq.(4) H_0 serves as the unperturbed Hamiltonian and H' as the perturbative part. It is easy to see that H_0 is diagonal and the eigen-wave functions for H_0 are[6]

$$
|{}^{\lambda}_{\mu\nu}\rangle = (-1)^{s+s_z}\sqrt{\lambda} D^{(\lambda)}_{\mu\nu}(A), \qquad \mu = ({}^{IY}_{I_3}), \qquad \nu = ({}^{s\ 1}_{-s_z}). \tag{7}
$$

$|{}^{8}_{\mu\nu}\rangle$ and $|{}^{10}_{\mu\nu}\rangle$ correspond to baryon octet and baryon decuplet respectively. As one computes the matrix elements of $\langle H'\rangle$ a useful formula is as follows:

$$
\langle{}^{\lambda_2}_{\mu_2\nu_2}|D^{(\lambda)}_{\mu\nu}(A)|{}^{\lambda_1}_{\mu_1\nu_1}\rangle = (-1)^{s_1+s_{1z}+s_2+s_{2z}}\sqrt{\frac{\lambda_1}{\lambda_2}}\sum_{\gamma}\begin{pmatrix}\lambda_1 & \lambda & \lambda_{2\gamma}\\ \mu_1 & \mu & \mu_2\end{pmatrix}\begin{pmatrix}\lambda_1 & \lambda & \lambda_{2\gamma}\\ \nu_1 & \nu & \nu_2\end{pmatrix} \tag{8}
$$

with a standard notation for the Clebsch-Gordan (CG) coefficients. Here γ distinguishes the independent irreducible representations occurring in the reduction $(\lambda)\otimes(\lambda_1)\rightarrow(\lambda_2)$. The mass of the baryon for $|k\rangle \equiv |{}^{\lambda}_{\mu\nu}\rangle$ can be calculated in perturbation

$$
M_k = E_k^{(0)} + E_k^{(1)} + E_k^{(2)} + \cdots, \tag{9}
$$

where

$$
\begin{aligned}
E_k^{(0)} &= \langle k|H_0|k\rangle, && (10)\\
E_k^{(1)} &= \langle k|H'|k\rangle, && (11)\\
E_k^{(2)} &= \sum_{n\neq k}\frac{|\langle n|H'|k\rangle|^2}{E_k^{(0)} - E_n^{(0)}} && (12)
\end{aligned}
$$

$E_k^{(0)}$ and $E_k^{(1)}$ have been known in literatures[6]. Our object is to calculate $E_k^{(2)}$ which is related to non-diagonal matrix elements of $D^{(8)}_{8,8}(A)$ between the baryon states (see Eq.(12)). We will study the cases of baryon octet and baryon decuplet respectively.

1) The baryon-octet case: Noticing SU(3)-multiplets decomposition formula

$$
8\otimes 8 = 27 \oplus 10^* \oplus 8_F \oplus 8_D \oplus 1, \tag{13}
$$

we have (see Eq.(12)) $|n\rangle \in \{27,\ 10,\ 10^*,\ 1\}$. AS $|k\rangle \in$ nucleon, the nonzero $(n \neq k)$ $\langle n|D^{(8)}_{8,8}|k\rangle-$matrix elements are follows:

$$
\begin{aligned}
\langle{}^{(10^*)}_{\mu\nu}|D^{(8)}_{({}^{00}_{0})({}^{00}_{0})}|{}^{(8)}_{({}^{\frac{1}{2}1}_{\frac{1}{2}})({}^{\frac{1}{2}1}_{\frac{1}{2}})}\rangle &= \sqrt{\frac{1}{20}}, && (14)\\
\langle{}^{(27)}_{\mu\nu}|D^{(8)}_{({}^{00}_{0})({}^{00}_{0})}|{}^{(8)}_{({}^{\frac{1}{2}1}_{\frac{1}{2}})({}^{\frac{1}{2}1}_{\frac{1}{2}})}\rangle &= \sqrt{\frac{3}{50}},
\end{aligned}
$$

here equation (8) and the SU(3) CG coefficients listed in Ref.[7] have been used. Since the spins of $|k\rangle$ and $|n\rangle$ are same, from Eqs (5) and (10) we have

$$
E_k^{(0)} - E_n^{(0)} = \frac{1}{2}b^{-2}[C_2(k) - C_2(n)], \tag{15}
$$

where $C_2(k)$ denotes the Casimir operator for the k-dimensional irreducible representation of SU(3). Noticing $C_2(8) = 3,\ \ C_2(10^*) = 6$ and $C_2(28) = 8$, then

$$\begin{aligned} E_8^{(0)} - E_{10^*}^{(0)} &= -\frac{3}{2}b^{-2}, \\ E_8^{(0)} - E_{27}^{(0)} &= -\frac{5}{2}b^{-2}, \end{aligned} \tag{16}$$

Combining the above equations with the known results in Refs[5] and [6], we have the nucleon's mass to the second order in perturbation

$$M_N = M_8 - \frac{3}{10}m - \frac{43}{750}g, \tag{17}$$

where $M_s = \langle H_0\rangle_{\lambda=8} + m,\ \ g = m^2b^2$. Through the similar computations, we can obtain the masses of $\Lambda,\ \ \Sigma$ and Ξ to the second order in perturbation

$$\begin{aligned} M_\Lambda &= M_8 - \frac{1}{10}m - \frac{9}{250}g, \\ M_\Sigma &= M_8 + \frac{1}{10}m - \frac{37}{750}g, \\ M_\Xi &= M_8 + \frac{1}{5}m - \frac{3}{125}g, \end{aligned} \tag{18}$$

where $M_8,\ m$ and g are model-dependent variables. Eliminating $M_8,\ m$ and g in Eqs (17) and (18), we have

$$2(M_N + M_\Xi) = 3M_\Lambda + M_\Sigma + \frac{2}{13}(M_N + M_\Sigma - 2M_\Lambda). \tag{19}$$

Comparing Eq.(19) with Eq.(1), we can see that the last term on the right-hand side of Eq.(19) serves as a correction coming from the skyrmion dynamics to GOR in the octet case. Thus we get the desired result

$$\delta m_8 = \frac{2}{13}(M_N + M_\Sigma - 2M_\Lambda). \tag{20}$$

By using the mass relation of Eq.(19), the mass variable $(M_N,\ M_\Sigma,\ M_\Lambda)$ in Eq.(20) can be changed to be $(M_\Xi,\ M_\Sigma,\ M_\Lambda)$, or $(M_N,\ M_\Sigma,\ M_\Xi)$, or $(M_N,\ M_\Xi,\ M_\Lambda)$. Namely, we can also write δm_8 as

$$\begin{aligned} \delta m_8 &= \frac{1}{12}(3M_\Sigma - 2M_\Xi - M_\Lambda) \\ &= \frac{2}{35}(5M_\Sigma - M_N - M_\Xi) \\ &= \frac{1}{15}(6M_N + 4M_\Xi - 10M_\Lambda). \end{aligned} \tag{21}$$

These four expressions of δm_8 are equivalent to each other in principle. However, since equation (19) is still not an exact identity of masses when experimental mass data are inputs, the numerical results given by Eqs (20)–(21) we get

$$\delta m_8 = -15.35 \pm 1.32MeV.$$

This second order correction to δm_8 agrees with the experiment qualitatively, and in quantitative respect it is compatible with the data showing in Eq (3) roughly.

2) The baryon-decuplet case: As $|k\rangle$ (in EQ. (12)) belongs to decuplet, from the SU(3)-decomposition formula $10 \otimes 8 = 35 \oplus 27 \oplus 10 \oplus 8$, we have $|n\rangle \in \{35,\ 27,\ 8\}$. Using again the formulas (8) and (15), the CG coefficients of SU(3)[7] and $C_2(35) = 12$, we get the masses of the baryon decuplet to the second order in perturbation

$$M_\Delta = M_{10} - \frac{1}{8}m - \frac{85}{672}g, \qquad M_{\Sigma^*} = M_{10} - \frac{26}{336}g, \tag{22}$$

$$M_{\Xi^*} = M_{10} + \frac{1}{8}m - \frac{9}{224}g, \qquad M_\Omega = M_{10} + \frac{1}{4}m - \frac{5}{336}g, \tag{23}$$

where $M_{10} = \langle H_{10} \rangle_{\lambda=10}$. Here there are four equations and three unknowns. By solving these over-determinant equations, a constrained equation is left, which is

$$M_\Omega - M_\Delta = 3(M_{\Xi^*} - M_{\Sigma^*}). \tag{24}$$

Equation (24) is called the Okubo relationship which holds to the second order in flavor breaking as shown by Okubo long ago[8] and by Morpurgo recently[2]. Here, we reach such conclusion again in the skyrmion formalism. Equations (22)–(23) show that the equal spacing rule for the decuplet (i.e., GOR in decuplet case) no longer holds. Using the definitions of $\delta m_{10}^{(1)}$ and $\delta m_{10}^{(2)}$ (Eq.(2)), and the mass-splitting formulas (22)–(23), we have

$$\delta m_{10}^{(1)} = \frac{g}{84}, \qquad \delta m_{10}^{(2)} = \frac{g}{42}.$$

Then we get $\delta m_{10}^{(2)}/\delta m_{10}^{(1)} = 2$, which is in good agreement with the experiment (see Eqs.(3)) $(\delta m_{10}^{(2)}/\delta m_{10}^{(1)})_{\rm expt} = 2.1$. From Eqs (22)–(23), g can be found out to be $84(2M_{\Sigma^*} - M_{\Xi^8} - M_\Delta)$. Then we get $\delta m_{10}^{(1)} = 2M_{\Sigma^*} - M_{\Xi^8} - M_\Delta$. By using the Okubo relationship (24) the mass variables in $\delta m_{10}^{(1)} = 2M_{\Sigma^*} - M_{\Xi^8} - M_\Delta$ could be changed. The other three expressions for $\delta m_{10}^{(1)}$ read

$$\begin{aligned} \delta m_{10}^{(1)} &= \frac{1}{3}(3M_{\Xi^*} - 2M_\Omega - M_\Delta) \\ &= \frac{1}{3}(3M_{\Sigma^*} - 2M_\Delta - M_\omega) \\ &= 2M_{\Xi^*} - M_{\Sigma^*} - M_\Omega. \end{aligned} \tag{25}$$

These four $\delta M_{10}^{(1)}$ expressions are equivalent in principle, but their numerical results are not exactly same since equation (24) is approximate just like the above octet case. Thus we have

$$\delta m_{10}^{(1)} = 7.05 \pm 2.55 MeV, \qquad \delta m_{10}^{(2)} = 14.1 \pm 5.1 MeV,$$

where $\delta m_{10}^{(2)}/\delta m_{10}^{(1)} = 2$ has been used. They agree with the experimental data shown in Eqs (3).

The picture that the baryons are regarded as the chiral solitons has extensively been investigated during last ten years[9]. It is well known that the quantum mechanics of SU(3) skyrmion is a basic formalism to deal with $(ud)-$ and $s-$flavor breaking in the model. Following the investigations of the first order in flavor breaking in this quantum mechanics problem, we have completed the calculations of the second order corrections to the baryon-mass splitting and shown that the results support the soliton pictures. Especially, since high order in perturbations of quantum mechanics are intimately related to the H'-matrix elements between various eigenstates of H_0 (see Eqs (9)–(12)), our above results can be regarded as signals of the existence of some SU(3) rotation excitation states of baryons: 27-let (with spin $\frac{1}{2}$ or $\frac{3}{2}$, 10*-let (with spin $\frac{1}{2}$) and 35-let (with spin $\frac{3}{2}$. It is easy to be sure that the wavefunctions of these SU(3) states satisfy the constrained condition coming from the Wess-Zumino term in the QCD effective Lagrangian, i.e., the spin hypercharge $Y_R = 1$[5, 10]. So they should be physical states in QCD.

From Eq.(5) the kinetic energy of usual SU(2) rotations of skyrmion is $T(SU(2)) = [C_2(SU(2))]/(2I_2)$, $I_2 = (a^2b^2)/(a^2 - b^2)$, where I_2 is the moment of inertia for SU(2) rotation, and $C_2(SU(2))$ $(= \sum_{A=1}^{3} R_A R_A \equiv \hat{J}^2)$ is the Casimir operator for SU(2), or the square of total angular momentum operators. Similarly, again from Eq.(5), the kinetic energy of SU(3) rotations of skyrmion reads $T(SU(3)) = [C_2(SU(3))]/(2I_3)$, $I_3 = b^2$, where I_3 corresponds to the monment of inertia for SU(3) rotation, $C_2(SU(3))$ $(= \sum_{i=1}^{8} L_i L_i)$ is the Casimir operator for SU(3). Thus the mode of SU(3) rotation excitations can be understood as a natural extension of the usual SU(2) rotation excitations. Clearly, all of the quantum numbers for both SU(3) rotation excitation state (SU(3)RES) and its corresponding octet or decuplet baryons are the same except their SU(3) Casimir operators. By using 10_x^*, 27_x and 35_x to denote SU(3)RES of $x \in$ {octet, decuplet}, we may reasonably conclude that there exist 16 SU(3)RES which have visible contributions to the corrections to GOR, and they are 10_N^*, 10_Λ^*, 10_Σ^*, 10_Ξ^*, 27_N, 27_Λ, 27_Σ, 27_Ξ, 27_Δ, 27_{Σ^*}, 27_{Ξ^*}, 27_Ω, 35_Δ, 35_{Σ^*}, 35_{Ξ^*} and 35_Ω.

Finally, we would like to make two remarks. First, in the present paper the mass spectra of the octet case and the decuplet case calculated in perturbations respectively and independently. However, due to the limitations of perturbation method for the $(ud)-$ and $s-$flavor symmetry breaking, it cannot be extended to the case of the octet and the decuplet combined. If one wants to get a complete mass spectrum (including octet and decuplet simultaneously), the skyrmion quantum mechanics problem in the SU(3) collective coordinate space (see Eqs (4)–(6)) have to be solved exactly. Unfortunately, this goal can merely be reached numerically[10], instead of analytically. Therefore, in order to reveal the physical implications explicitly, our analytical analyses in the above discussions are

necessary. Second, our analyses on δm_8, $\delta m_{10}^{(1)}$, and $\delta m_{10}^{(2)}$ are model-independent. In other words, we did not use any concrete $\mathcal{L}_{eff}$ to do calculations till now, and all conclusions in this paper come from the studies on generic structure of the Hamiltonian of the SU(3) quantum mechanics (Eqs (4)–(6)). Thus our predictions on existence of SU(3)RES are also model-independent in the skyrmion formalism.

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